REINFORCEMENT LEARNING

A playful machine learning

Avneet Kaur
PhD Applied Mathematics
Email: a93kaur@uwaterloo.ca

GUESS THE ANIMAL

QUOLL

WHAT IS MACHINE LEARNING?

- machines learn to do a given task without being explicitly programmed.

Supervised learning	Unsupervised learning	Reinforcement learning
• Labelled dataset • Learn f to map y=f(x) • Classification, Regression	• Unlabelled dataset • Learn underlying structure • Clustering, Dimensionality reduction	• Generate dataset • Maximize utility by learning to interact • Robot navigation, learning games

TRANSLATE THESE WORDS

- ਕੰਨ (Punjabi)

- Nez (French)

KEY TAKEAWAYS

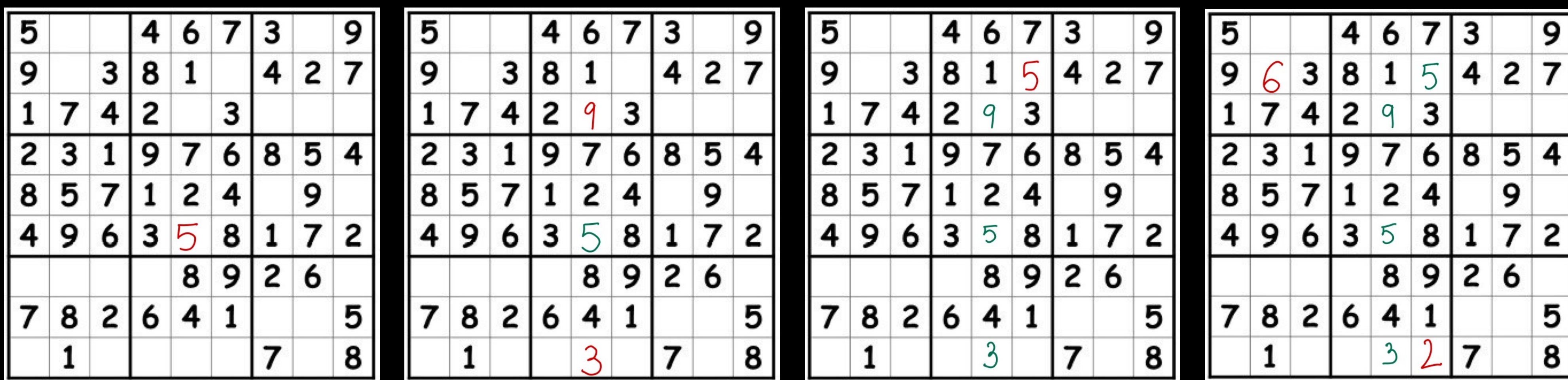
- You were rewarded for each type of answer.
- You as an agent interacted with the environment to translate better.
- Environment gave feedback in the form of rewards.

SUDOKU

5			4	6	7	3		9
9		3	8	1		4	2	7
1	7	4	2		3			
2	3	1	9	7	6	8	5	4
8	5	7	1	2	4		9	
4	9	6	3		8	1	7	2
				8	9	2	6	
7	8	2	6	4	1			5
	1					7		8

Task: Fill the missing squares in as less time as possible.

- Agent makes a sequence of moves(actions)
- Each move by the agent decides which subsequent squares can be filled next



- Reaching the goal state will have a reward
- Intermediate squares may or may not have reward

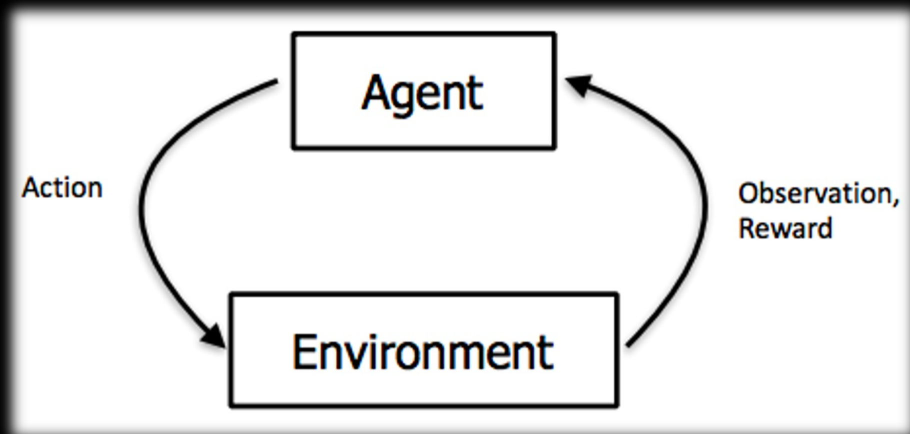
5			4	6	7	3		9
9	6	3	8	1	5	4	2	7
1	7	4	2	9	3			
2	3	1	9	7	6	8	5	4
8	5	7	1	2	4		9	
4	9	6	3	5	8	1	7	2
			7	8	9	2	6	
7	8	2	6	4	1			5
	1			3	2	7		8

An intermediate state

5	2	8	4	6	7	3	1	9
9	6	3	8	1	5	4	2	7
1	7	4	2	9	3	5	8	6
2	3	1	9	7	6	8	5	4
8	5	7	1	2	4	6	9	3
4	9	6	3	5	8	1	7	2
3	4	5	7	8	9	2	6	1
7	8	2	6	4	1	9	3	5
6	1	9	5	3	2	7	4	8

Goal state

RL FRAMEWORK



INVENTORY CONTROL EXAMPLE

- **Observation:** Stock level
- **Action:** What to purchase
- **Reward:** Profit

ENVIRONMENT

- An external system that an agent can perceive and act on
- Receives action from agent and in response emits appropriate reward and (next) observation

AGENT

- A system that takes actions to change the state of the environment (Decision maker)
- Executes action upon receiving observation
- For taking an action the agent receives an appropriate reward

STATE

- State can be viewed as a summary or an abstraction of the history of the system
- For example, in Sudoku, the state could be raw image or vector representation of the board

REWARD

- Reward is a scalar feedback signal
- Indicates how well agent acted at a certain time
- The agent’s aim is to maximise cumulative reward

COMPONENTS OF AN RL AGENT

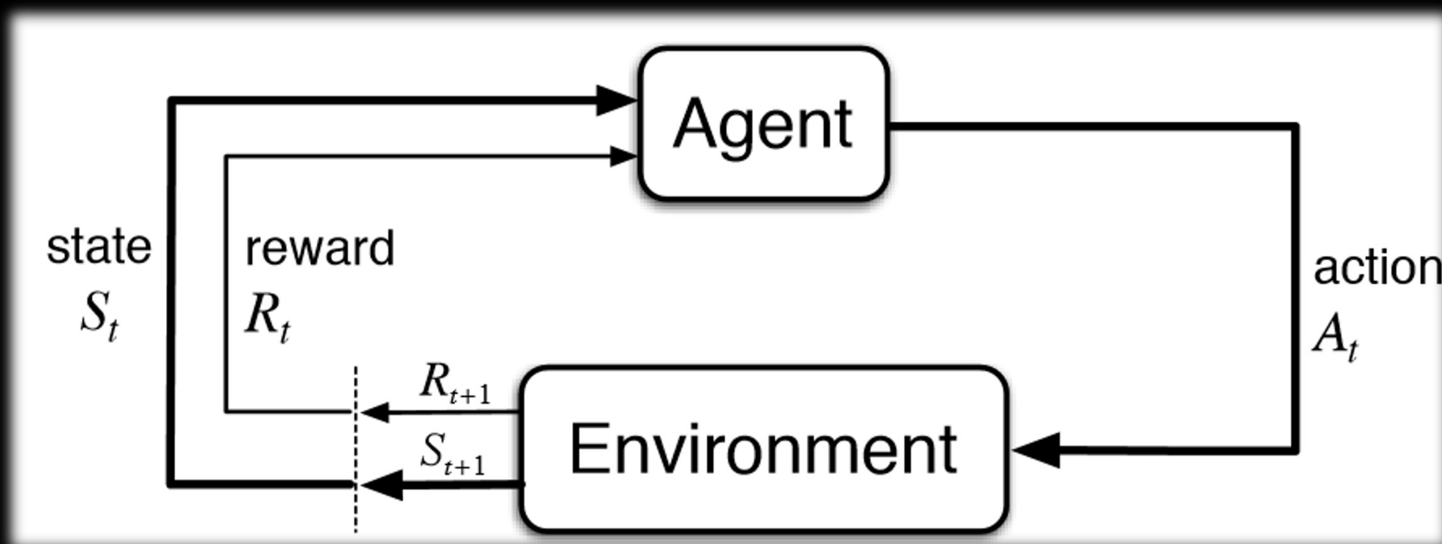
- **Policy:** agent's behaviour function; $\pi: S \rightarrow A$
- **Value function:** evaluates how good is each state and/or action. Therefore, it is used to choose appropriate action among the available options.
- **Model:** agent's representation of the environment; Mainly contains state transition information and reward function.

TIC TAC TOE

- **Observation:** Board position
- **Action:** Moves
- **Reward:** Win or loss
- **Policy:** Agent has multiple empty squares to choose
  - Random policy is to place 'X' in any one of empty squares randomly
  - Better policy is to place 'X' in square 5
- **Value Function:** Agent may have an estimate about the value of being in a certain board configuration
- **Model:** Model of transition probabilities between states

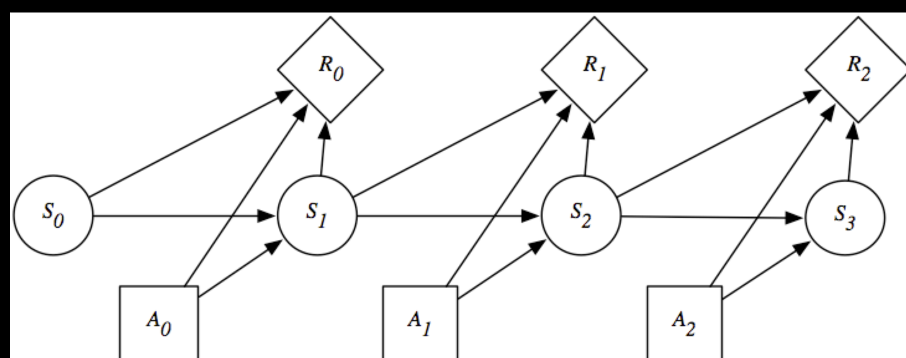
X_1	O_2	3
X_4	5	6
O_7	O_8	X_9

FRAMEWORK



MARKOV DECISION PROCESS

- Provides a mathematical framework for modelling decision making process
- Can formally describe the working of the environment and the agent
- Core problem in solving an MDP is to find an 'optimal' policy (or behaviour) for the decision maker (agent) to maximize the total future reward



RANDOM VARIABLE

- A random variable X denotes the outcome of a random phenomenon
- Examples include outcome of a coin toss and the roll of a dice.

STOCHASTIC PROCESS

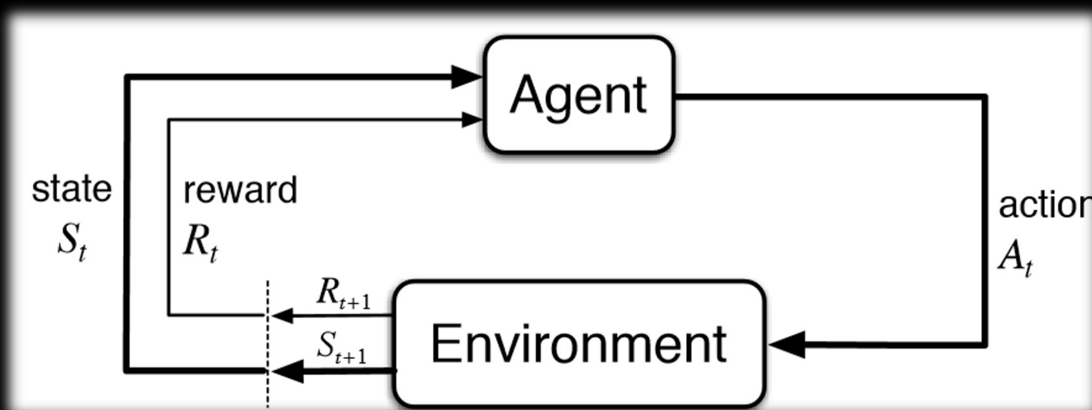
- It is a collection of random variables indexed by some mathematical set T.
- T has the interpretation of time and is typically, $\mathbb{N}$ or $\mathbb{R}$. Assume T=$\mathbb{N}$ for our sessions.
- Notation: $\{X_t\}_{t \in T}$

MARKOV PROPERTY

- A stochastic process $\{S_t\}_{t \in T}$ is said to have Markov property if for any state s_t,

$$P(S_{t+1} \mid S_t) = P(S_{t+1} \mid S_1, S_2, \ldots, S_t).$$

- S_t captures all relevant information from history and is a sufficient statistic of the future.
- Memoryless property



STATE TRANSITION PROBABILITY

- For a stochastic process $\{S_t\}_{t \in T}$, the state transition probability for successive states s and s′ is denoted by

$$\mathcal{P}_{SS'} = P(S_{t+1} = S' \,|\, S_t = S).$$

- State transition matrix $\mathcal{P}$ then denotes the transition probabilities from all states s to all successor states s′ (with each row summing to 1.

$$\mathcal{P} = \begin{pmatrix} \mathcal{P}_{11} & \mathcal{P}_{12} & \cdots & \mathcal{P}_{1n} \\ . & & & \\ . & & & \\ . & & & \\ \mathcal{P}_{n1} & \mathcal{P}_{n2} & \cdots & \mathcal{P}_{nn} \end{pmatrix}$$

MARKOV CHAIN

- A stochastic process $\{s_t\}_{t \in T}$ is a Markov Chain if it satisfies Markov property.
- It is represented by the tuple $< \mathfrak{S}, \mathscr{P} >$ where $\mathfrak{S}$ denotes the set of states.
- It is also called Markov process.

$$P[S_{t+1} | S_t] = P[S_{t+1} | S_1, \ldots, S_t]$$

MARKOV REWARD PROCESS

A Markov reward process is a tuple < $\mathcal{S}, \mathcal{P}, \mathcal{R}, \gamma$ > is a Markov chain with values

- $\mathcal{S}$: Finite set of states
- $\mathcal{P}$: State transition probability
- $\mathcal{R}$: Reward for being in state s_t is given by a deterministic function $\mathcal{R}$

$$r_{t+1} = \mathcal{R}(s_t)$$

- γ: Discount factor such that $\gamma \in [0,1]$

WHY DISCOUNTING?

- Offers trade off between ‘myopic’ and ‘far sighted’ rewards
- Avoids infinite returns in cyclic and infinite horizon Markov processes
- Undiscounted Markov reward process are mostly used when sequences terminate.

TOTAL DISCOUNTED REWARD

Total discounted reward from time step t is, $\sum_{k=0}^{\infty}(\gamma^k r_{t+k+1})$

- $\gamma \rightarrow 0$ (myopic); $\gamma \rightarrow 1$ (far-sighted)
- Value of reward r after k+1 timesteps is γ^kr.

STATE-VALUE FUNCTION

Value function V(s) denotes the long-term value of state s,

$$V(s) = \mathbb{E}(G_t | s_t = s) = \mathbb{E}(\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} | s_t = s)$$

and is independent of time, t.

RECURSIVE FORMULATION OF VALUE FUNCTION

$$V(s) = \mathbb{E}(G_t | s_t = s) = \mathbb{E}(\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} | s_t = s)$$

$$= \mathbb{E}(r_{t+1} + \gamma r_{t+2} + \gamma^2 r_{t+3} + \cdots | s_t = s)$$

$$= \mathbb{E}(r_{t+1} | s_t = s) + \gamma \mathbb{E}(G_{t+1} | s_t = s)$$

$$= \mathbb{E}(r_{t+1} | s_t = s) + \gamma \mathbb{E}(\mathbb{E}(G_{t+1} | s_t = s) | s_t = s)$$

$$= \mathbb{E}(r_{t+1} | s_t = s) + \gamma \mathbb{E}(V(s_{t+1}) | s_t = s)$$

$$= \mathbb{E}(r_{t+1} + \gamma V(s_{t+1}) | s_t = s)$$

BELLMAN EQUATION FOR MRP

- For $s' \in \mathfrak{S}$, a successor state of s with transition probability $\mathcal{P}_{ss'}$, we can rewrite $V(s)$ as

$$V(s) = \mathbb{E}(r_{t+1}) + \gamma \sum_{s' \in \mathfrak{S}} \mathcal{P}_{ss'} \mathrm{V}(\mathrm{s}').$$

- This is the Bellman equation for value functions

MATRIX FORM

- Let $\mathcal{S}$={1,2,…n} and $\mathcal{P}$ be known. Then,

$$V = \mathcal{R} + \gamma \mathcal{P} V$$

where

$$\begin{bmatrix} V(1) \\ V(2) \\ . \\ . \\ . \\ V(n) \end{bmatrix} = \begin{bmatrix} \mathcal{R}(1) \\ \mathcal{R}(2) \\ . \\ . \\ . \\ \mathcal{R}(n) \end{bmatrix} + \gamma \begin{bmatrix} \mathcal{P}_{11} & \cdots & \mathcal{P}_{1n} \\ \mathcal{P}_{21} & \cdots & \mathcal{P}_{2n} \\ \vdots & \ddots & \vdots \\ \mathcal{P}_{n1} & \cdots & \mathcal{P}_{nn} \end{bmatrix} \times \begin{bmatrix} V(1) \\ V(2) \\ . \\ . \\ . \\ V(n) \end{bmatrix}.$$

Solving for V, we get,

$$V = (I - \gamma \mathcal{P})^{-1} \mathcal{R}.$$

ABSORBING STATE

- A state $i \in \mathfrak{I}$ is said to be absorbing if it is impossible to leave that state, Mathematically,

$$P_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

- In a game of snake and ladders, the state '100' is an absorbing state.

MARKOV DECISION PROCESS

MDP is a tuple $<\mathcal{S},\mathcal{A},\mathcal{P},\mathcal{R},\gamma>$ where

- $\mathcal{S}$: Finite set of states
- $\mathcal{A}$: Finite set of actions
- $\mathcal{P}$: State transition probability

$$\mathcal{P}_{ss'}^{a} = \mathbb{P}(s_{t+1} = s' | s_t = s, a_t = a), a_t \in A$$

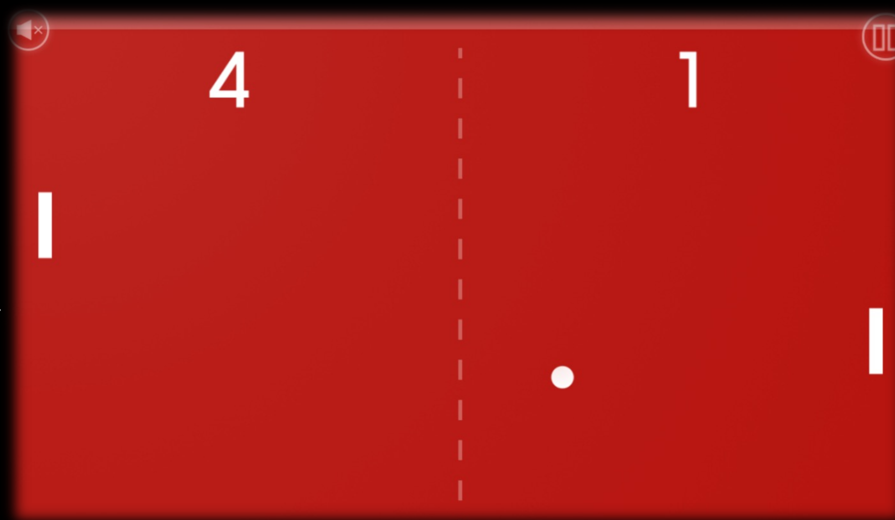
- $\mathcal{R}$: Reward for taking action a_t at state s_t and transitioning to state s_{t+1} is given by the deterministic function $\mathcal{R}$
- $$r_{t+1} = \mathcal{R}(s_t, a_t, s_{t+1}).$$
- γ: Discount factor such that $\gamma \in [0,1]$

SNAKE AND LADDERS

States: Each square from 1 to 100
Actions: Move right, climb ladders, or come down snakes depending on the number on the die throw
Rewards: -1 for every move made until reaching '100'

ATARI-PONG GAME

States: Possible set of all images
Actions: Paddle up or down
Rewards:
+1 for making the opponent miss the ball,
-1 if the agent misses the ball,
0 otherwise.



POLICY

- Let π denote a policy that maps state space $\mathfrak{S}$ to action space $\mathscr{A}$.

There are 2 types of policies:

- Deterministic policy: $a=\pi(s)$, $s \in \mathfrak{S}$, $a \in \mathscr{A}$.
- Stochastic policy: $\pi(a|s)=P[a_t=a|s_t=s]$

TIC TAC TOE REVISITED

- **Deterministic Policy:** Place 'X' in square 5
- **Stochastic policy:** Place 'X' in square 5 with probability 0.8 and place 'X' in square 6 with probability 0.2

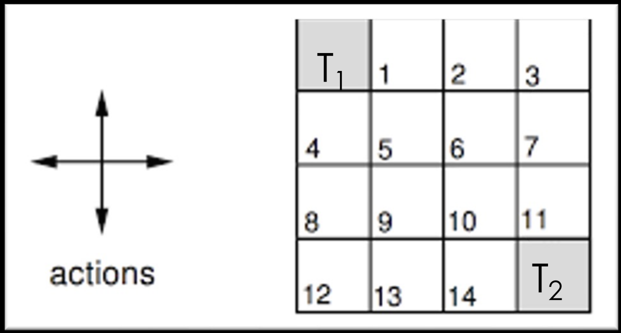
X_1	O_2	3
X_4	5	6
O_7	O_8	X_9

NAVIGATION GRID

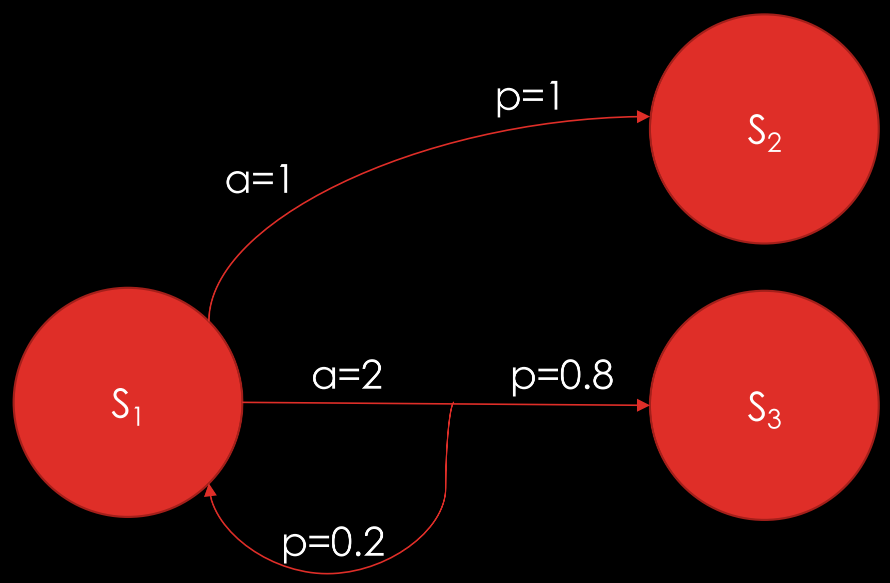
- States: $\{1, 2, \ldots, 14, T_1, T_2\}$
- Actions: {right, left, up, down}
- Deterministic Policy:

$$\pi(s) = \begin{cases} down, & s = \{3,7,11\} \\ right, & otherwise \end{cases}$$

- Example sequences: $\{\{12,13,14,T_2\}, \{4,5,6,7,11,T_2\}\}$
- Stochastic Policy: $\pi(a|s)$ could be a uniform random action between all possible actions at state s
- Example sequences: {{4,5,9,8,12,…},{1,2,6,5,1,2,3,…}}



GRAPHICAL NOTATION



VALUE FUNCTION REVISITED

- The value function $V(s)$ under policy π in state s is the expected return starting from state s and then following policy π

$$V^{\pi}(s) = \mathbb{E}_{\pi}\left(\sum_{k=0}^{\infty} \gamma^k r_{t+k+1} | s_t = s\right) = \mathbb{E}_{\pi}(r_{t+1} + \gamma V^{\pi}(s_{t+1}) | s_t = s).$$

- Goal: Find policy π that maximizes $V^{\pi}(s)$.

EXAMPLE

S_1	S_2	S_3	S_4
(0)	(0)	(0)	(100)

Compute $V(s_2)$ and $V(s_3)$ with $\gamma = 1$

- Policy 1: Move left or right with equal probability
  Solution: $V(s_2) = 0 * 0.5 + 0 * 0.5 = 0$
  $V(s_3) = 0 * 0.5 + 100 * 0.5 = 50$
- Policy 2: Move left or right with probability 0.6 and 0.4 respectively
  Solution: $V(s_2) = 0 * 0.6 + 0 * 0.4 = 0$
  $V(s_3) = 0 * 0.6 + 100 * 04 = 40$
- Policy 3: Move right with probability 1
  Solution: $V(s_2) = 0 * 1 = 0$
  $V(s_3) = 100 * 1 = 100$

ACTION-VALUE FUNCTION

- The action value function $Q(s,a)$ under policy π is the expected return starting from state s and taking action a following policy π
$$Q^{\pi}(s,a) = \mathbb{E}_{\pi}\left(\sum_{k=0}^{\infty} \gamma^{k} r_{t+k+1} \,\middle|\, s_t = s, a_t = a\right).$$
- It can be decomposed as
$$Q^{\pi}(s,a) = \mathbb{E}_{\pi}\left(r_{t+1} + \gamma Q^{\pi}(s_{t+1,} a_{t+1}) \middle| s_t = s, a_t = a\right).$$

RELATIONSHIP BETWEEN V^{π} AND Q^{π}

$$V^{\pi}(s) = \sum_{a \epsilon \mathcal{A}} \pi(a|s) Q^{\pi}(s, a)$$

OPTIMAL VALUE FUNCTION

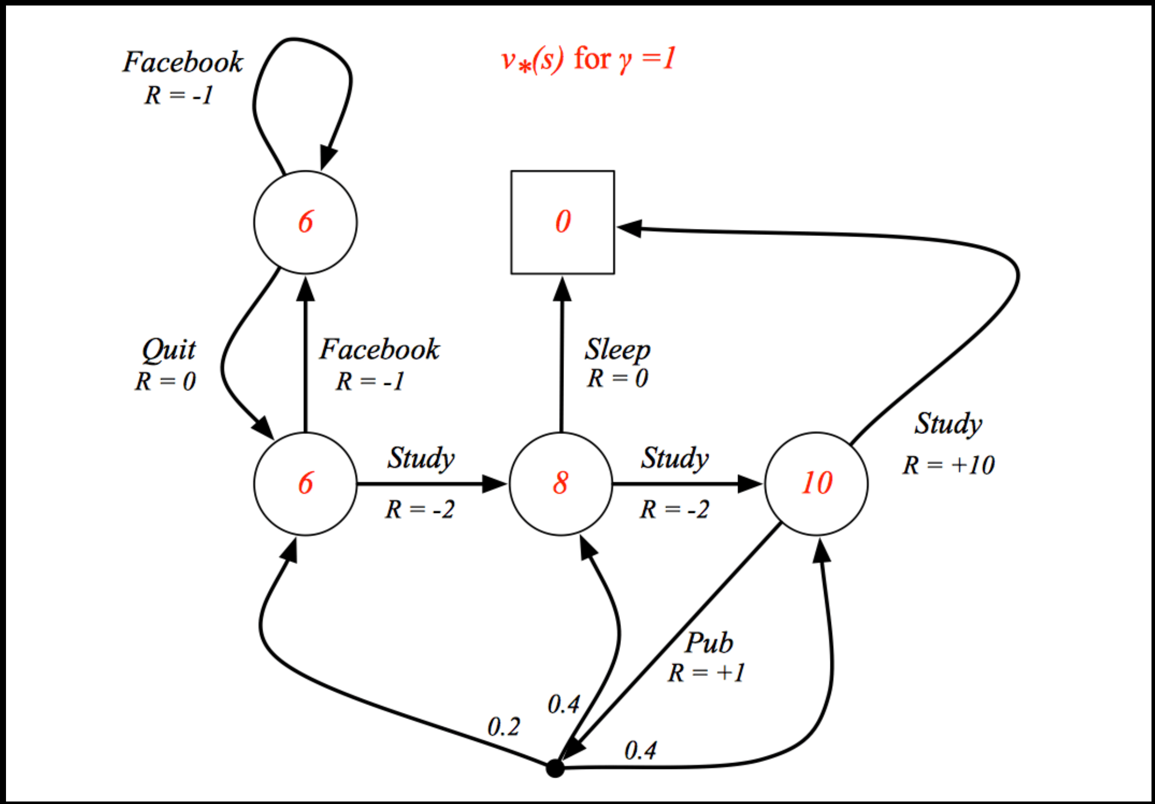
- The optimal value function V_*, for state s, is the maximum value function over all policies.
- Mathematically,

$$V_*(s) = \max_{\pi \epsilon \Pi_{stat}} V^{\pi}(s).$$

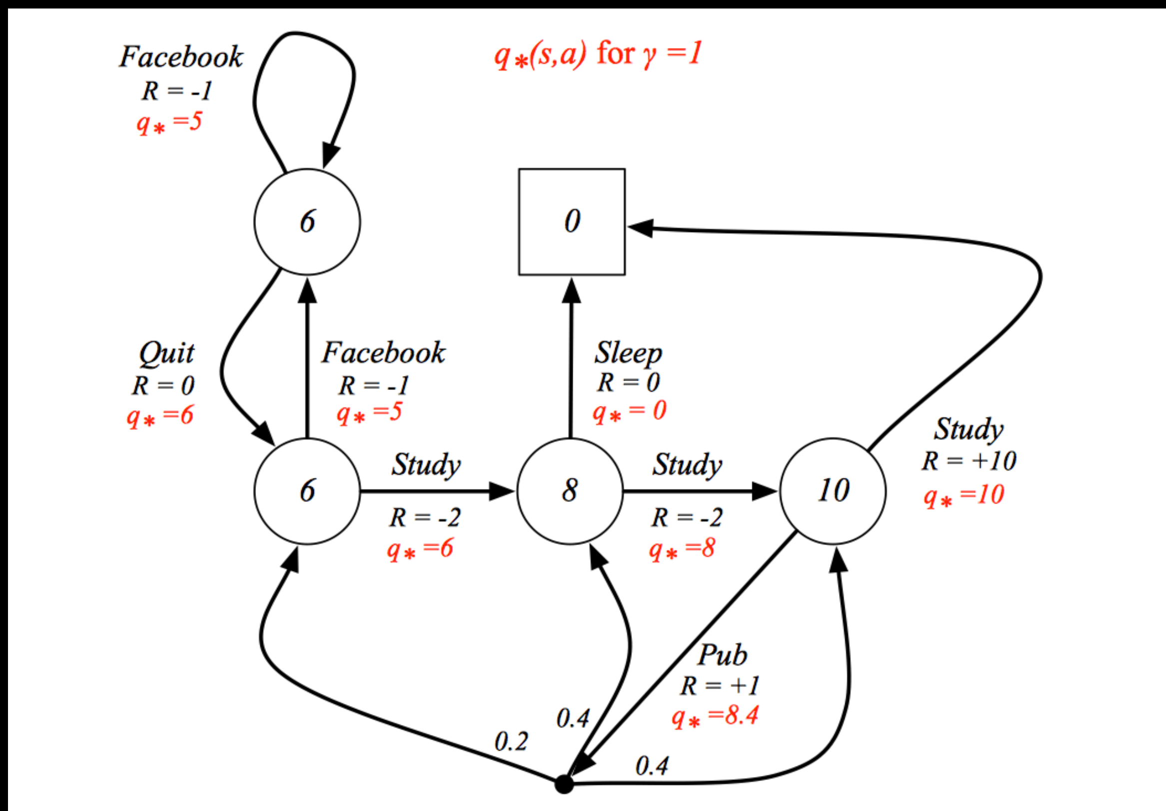
- The optimal action-value function $Q_*(s,a)$, for a state s and action a, is the maximum action-value function over all policies.
- Mathematically,

$$Q_*(s,a) = \max_{\pi \epsilon \Pi_{stat}} Q^{\pi}(s,a).$$

FOCUS EXAMPLE



FOCUS EXAMPLE CONTINUED



OPTIMAL POLICY

- Define a partial ordering of policies

$$\pi \geq \pi', \qquad if \quad V^{\pi}(s) \geq V^{\pi'}(s), \qquad \forall s \, \epsilon \, \mathfrak{S}$$

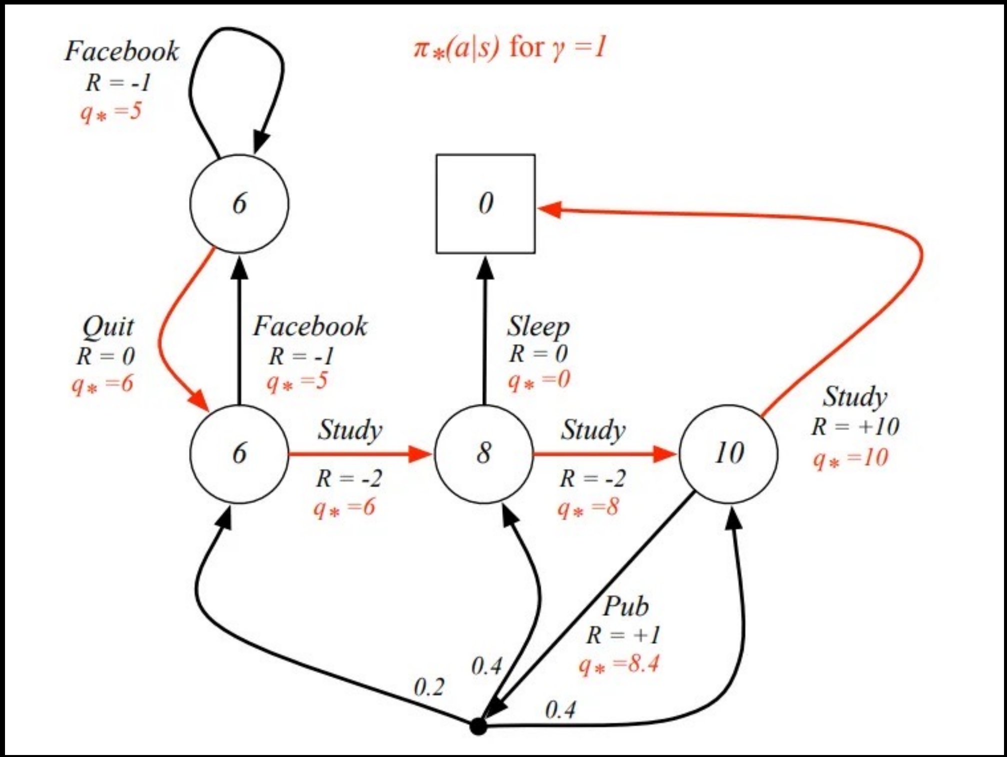
- There exists an optimal policy π_* that is better than or equal to all other policies.
- All optimal policies achieve the optimal value function,

$$V_*(s) = V^{\pi*}(s).$$

- All optimal policies achieve the optimal action-value function,

$$Q_*(s, a) = Q^{\pi*}(s, a)$$

FOCUS EXAMPLE REVISITED



RELATIONSHIP BETWEEN $V_*(\cdot) and\ Q_*(\cdot,\cdot)$

$$V_*(s) = \max_{a \in \mathcal{A}} Q_*(s, a)$$

GREEDY POLICY

- For any given $V(\cdot)$, define $\pi(a|s)$ as follows:

$$\pi^g = \pi(a|s) = greedy(V) = \begin{cases} 1, & if\ a = \arg\max_{a \epsilon \mathcal{A}} [\sum_{s' \epsilon \mathcal{S}} \mathcal{P}^a_{ss'}(\mathcal{R}^a_{ss'} + \gamma V(s'))] \\ 0, & otherwise \end{cases}$$

- For given $Q(\cdot,\cdot)$, define $\pi(a|s)$ as follows:

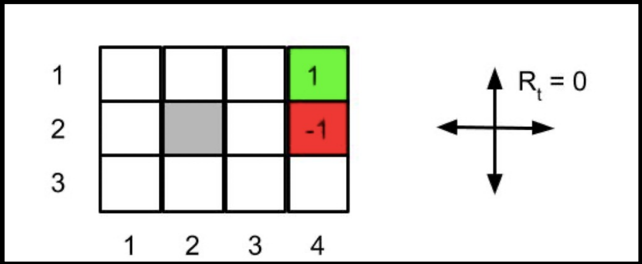
$$\pi^g = \pi(a|s) = greedy(V) = \begin{cases} 1, & if\ a = \arg\max_{a \epsilon \mathcal{A}} Q(s,a) \\ 0, & otherwise \end{cases}$$

- Greedy policy w.r.t. optimal (action) value function is an optimal policy.

NAVIGATION GRID

Case 1: Actions are successful(deterministic environment); $\gamma = 1$

- $V_*(1,4) = 1$
- $V_*(1,3) = 1$
- $V_*(1,2) = 1$
- $V_*(2,4) = -1$



Case 2: Actions are successful(deterministic environment); $\gamma = 0.9$

- $V_*(1,4) = 1$
- $V_*(1,3) = 0.9$
- $V_*(1,2) = 0.9^2$
- $V_*(2,4) = -1$

Case 3: Actions are successful with probability 0.8(stochastic environment);
With probability 0.1 each, you can go up and down; $\gamma = 0.9$

- $V_*(1,4) = 1$
- $V_*(1,3) = (0.8 * 0.9 * 1) + \big(0.1 * 0.9 * V_*(1,3)\big) + (0.1 * 0.9 * V_*(2,3))$
- Computation of $V_*(s)$ is not straightforward in such cases.

REFERENCES

1. Professor Easwer Subramaniyam's course CS5500 at IIT, Hyderabad,India
2. Prof. David Silver's RL course from Youtube(DeepMind)
3. Reinforcement Learning and Optimal Control by Dmitri Bertsekas
4. Prof. Pascal Poupart's course CS885 on Youtube(@pascalpoupart3507)